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# McGuire–Ruffini Solution in New General Relativity

**Mainuddin Ahmed[1] and Syed Moazzam Hossain[1]**

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We obtain a solution of new general relativity from a solution of Einstein's general relativity which includes many known solutions, such as Kerr–Newman–Kasuya, Kerr–Newman, Kerr, and NUT, as special cases.

## 1. INTRODUCTION

Hayashi and Nakano (1967) and Hayashi and Shirafuji (1979) obtained a new gravitational theory which is usually known as new general relativity (NGR). A feature of this NGR is absolute parallelism, the notion of which was first introduced by Einstein (1928a,b; 1929a,b; 1930a,b). Thus the spacetime of NGR is the Weitzenbock spacetime characterized by the metricity condition and by the vanishing of the curvature tensor. NGR describes all the observed gravitational phenomena observed in general relativity (GR). The Schwarzschild solution, Reissner–Nordstrom solution, and Weyl solution in GR are known in NGR (Hayashi and Shirafuji, 1979). According to Fukui and Hayashi (1981), stationary axially symmetric solutions of NGR are different from those of general relativity, but they have no explicit solution. On the other hand, recently stationary axially symmetric solutions of GR such as the Kerr and Kerr–Newman solutions have also been found in NGR (Toma, 1991; Kawai and Toma, 1992).

We believe that not only can Kerr and Kerr–Newman solutions of GR be found in NGR, but also the other solutions of GR which are not black hole solutions but have the common feature with the black hole solutions that they have horizons can be found in NGR.

[1]Department of Mathematics, Rajshahi University, Rajshahi-6205, Bangladesh.

In GR, a solution endowed with angular momentum, mass, and electric and electromagnetic and gravitational magnetic monopole parameters is known (McGuire and Ruffini, 1975). Hereafter this solution will be called the McGuire–Ruffini (MR) solution. The gravitational magnetic monopole parameter in MR solutions has been referred to as the NUT parameter (Demianski and Newman, 1966).

The MR solution is not a black hole solution. But it includes Kerr–Newman black hole solutions as a special case. It also includes the NUT solution as a special case which possesses very interesting properties. Although the MR solution is not a black hole solution, it has the common feature with the black hole solutions that it has horizon.

In this paper, we obtain a solution of NGR from the MR solution in GR.

## 2. BASIC FORMULATION OF NGR

In NGR, the fundamental fields of gravitation are the parallel vector fields

$$b_k = b_k^\mu \frac{\partial}{\partial x^\mu}$$

characterized by

$$D_\nu^* b_k^\mu = \partial_\nu b_k^\mu + \Gamma_{\lambda\nu}^{*\mu} b_k^\lambda = 0 \tag{2.1}$$

where

$$\Gamma_{\lambda\nu}^{*\mu} = b_k^\mu \partial_\nu b_\lambda^k \tag{2.2}$$

are the affine connection coefficients. The components of the metric tensor

$$g = g_{\mu\nu}\, dx^\mu \otimes dx^\nu$$

are given by

$$g_{\mu\nu} = b_\mu^k \eta_{kl} b_\nu^l \tag{2.3}$$

with

$$(\eta_{kl}) = \mathrm{diag}(-, +, +, +)$$

The gravitational Lagrangian in NGR is constructed with the torsion tensor

$$T_{\mu\nu}^\lambda = b_k^\lambda (b_\nu b_\mu^k - \partial_\mu b_\nu^k) \tag{2.4}$$

In the units such that $\hbar = c = 1$ the Lagrangian

$$L = -\frac{1}{3\kappa} (t^{\mu\nu\lambda} t_{\mu\nu\lambda} - v^\mu v_\mu) + \xi a^\mu a_\mu \tag{2.5}$$

used in this paper is quite in conformity with experiment (Hayashi and Nakano, 1967).

In (2.5), $\kappa$ is the Einstein gravitational constant, $\xi$ is a real, constant parameter, and $t_{\mu\nu\lambda}$, $v_\mu$, and $a_\mu$ are irreducible components of the torsion tensor:

$$t_{\mu\nu\lambda} = \frac{1}{2}(T_{\mu\nu\lambda} + T_{\nu\mu\lambda}) + \frac{1}{6}\left(g_{\lambda\mu}v_\nu + g_{\lambda\nu}v_\mu - \frac{1}{3}g_{\mu\nu}v_\lambda\right) \tag{2.6}$$

$$v_\mu = T^\lambda{}_{\lambda\mu} \tag{2.7}$$

$$a_\mu = \frac{1}{6}\epsilon_{\mu\nu\rho\sigma}T^{\nu\rho\sigma} \tag{2.8}$$

Here $\epsilon_{\mu\nu\rho\sigma}$ is the totally antisymmetric tensor normalized to $\epsilon_{0123} = -\sqrt{-g}$ with $g = \det(g_{\mu\nu})$.

The electromagnetic Lagrangian density is given by

$$L_{\mathrm{em}} = -\frac{1}{4}g^{\mu\rho}g^{\nu\sigma}F_{\mu\nu}F_{\rho\sigma} \tag{2.9}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{2.10}$$

Here $A_\mu$ is the electromagnetic vector potential.

The gravitational and electromagnetic field equations described by $L_G + L_{\mathrm{em}}$ are given by

$$G^{\mu\nu}(\{\cdot\}) + K^{\mu\nu} = \kappa T^{\mu\nu} \tag{2.11}$$

$$\partial_\mu(\sqrt{-g}J^{ij\mu}) = 0 \tag{2.12}$$

$$\partial_\nu(\sqrt{-g}F^{\mu\nu}) = 0 \tag{2.13}$$

$G^{\mu\nu}(\{\cdot\})$ in (2.11) is the Einstein tensor:

$$G_{\mu\nu}(\{\cdot\}) = R_{\mu\nu}(\{\cdot\}) - \frac{1}{2}g_{\mu\nu}R(\{\cdot\}) \tag{2.14}$$

where

$$R_{\mu\nu}(\{\cdot\}) = R^\rho{}_{\mu\rho\nu}(\{\cdot\}) = \partial_\mu\{\sigma^\rho\nu\} - \partial_\nu\{\sigma^\rho_\mu\} + \{\tau^\rho\mu\}\{\sigma^\tau\nu\} - \{\tau^\rho\nu\}\{\sigma^\tau_\mu\} \tag{2.15}$$

with

$$\{\sigma^\rho\nu\} = \frac{1}{2} g^{\rho\lambda}\{\partial_\sigma g_{\nu\lambda} + \partial_\nu g_{\sigma\lambda} - \partial_\lambda g_{\sigma\nu}\} \tag{2.16}$$

and

$$R(\{\cdot\}) = g^{\mu\nu}R_{\mu\nu}(\{\cdot\}) \tag{2.17}$$

$T^{\mu\nu}$ in (2.11) is the energy-momentum tensor given by

$$T^{\mu\nu} = F^{\mu\rho}F^{\nu\sigma}g_{\rho\sigma} + g^{\mu\nu}L_{\text{em}} \tag{2.18}$$

The tensors $K^{\mu\nu}$ in (2.11) and $J^{ij\mu}$ in (2.12) are defined by

$$\begin{aligned} K^{\mu\nu} = \frac{\kappa}{\lambda}\Bigg[ & \frac{1}{2}\{\epsilon^{\mu\rho\sigma\lambda}(T^{\nu}_{\rho\sigma} - T^{\nu}_{\rho\sigma}) \\ & + \epsilon^{\nu\rho\sigma\lambda}(T^{\mu}_{\rho\sigma} - T^{\mu}_{\rho\sigma})\}a_\lambda \\ & - \frac{3}{2}a^\mu a^\nu - \frac{3}{4}g^{\mu\nu}a^\lambda a_\lambda\Bigg] \end{aligned} \tag{2.19}$$

and

$$J^{ij\mu} = -\frac{3}{2} b^i_\mu b^j_\sigma \epsilon^{\rho\sigma\mu\nu}a_\nu \tag{2.20}$$

where

$$\lambda = \frac{4}{9}\xi + \frac{1}{3} \tag{2.21}$$

## 3. AN EXACT SOLUTION

Having now completed the preliminaries in Section 2, we now turn our attention to obtaining an exact solution of the field equations given by (2.11)–(2.13).

We seek a solution which will satisfy the conditions

$$b^k_\mu = \delta^k_\mu + \frac{a}{2} l^k l_\mu - \frac{e^2}{2} m^k m_\mu \tag{3.1}$$

and

$$g^{\mu\rho}g^{\nu\rho}F_{\rho\sigma} = \eta^{\mu\rho}\eta^{\nu\sigma}F_{\rho\sigma} \tag{3.2}$$

where $a$ and $e^2$ are arbitrary constant parameters and $l_\mu$ and $m_\mu$ are quantities satisfying the relations

$$\eta^{\mu\nu}l_\mu l_\nu = 0, \qquad \eta^{\mu\nu}m_\mu m_\nu = 0, \qquad \eta^{\mu\nu}l_\mu m_\nu = 0 \tag{3.3}$$

and $l^k$ and $m^k$ are defined as

$$l^k = \delta^k_\mu \eta^{\mu\nu} l_\nu, \qquad m^k = \delta^k_\mu \eta^{\mu\nu} m_\nu \tag{3.4}$$

By $\eta^{\mu\nu}$, $\eta_{\mu\nu}$ we will raise and lower the indices in $l_\mu$ and $m_\mu$ and by $\eta^{kl}$, $\eta_{kl}$ in $l^k$ and $m^k$.

We have from equation (3.3)

$$m_\mu = \zeta l_\mu \tag{3.5}$$

with $\zeta$ being a function of $x$. We also have from equation (3.2)

$$F_{\mu\nu}m^\nu = Am_\mu \tag{3.6}$$

with $A$ being a function of $x$. From equations (2.4) and (3.1), we have

$$T_{\lambda\mu\nu} = a\partial_{[\nu}(l_{\mu]}l_\lambda) - e^2\partial_{[\nu}(m_{\mu]}m_\lambda) \tag{3.7}$$

which gives the vanishing axial vector part

$$a_\mu = 0 \tag{3.8}$$

Then equation (2.11) reduces to

$$G^{\mu\nu}(\{\cdot\}) = \kappa T^{\mu\nu} \tag{3.9}$$

Equation (3.9) is identical with the Einstein equation in GR and equation (2.12) is trivially satisfied. Equation (2.13) reduces to

$$\partial_\nu F^{\mu\nu} = 0 \tag{3.10}$$

as $g = -1$ in the present case.

Following the same method as that in Kawai and Toma (1992), we have

$$(l_\mu) = \sqrt{a}(1, \lambda_1, l_2, \lambda_3) \tag{3.11}$$

$$(A_\mu) = -\frac{\kappa^q}{4\pi}\alpha(1, \lambda_1, \lambda_2, \lambda_3) \tag{3.12}$$

$$\zeta^2\alpha = \xi\xi^* \tag{3.13}$$

$$a = \frac{\kappa M}{4\pi}, \qquad e = \frac{q}{4\pi}\left(\frac{\kappa}{2}\right)^{1/2} \tag{3.14}$$

where $M$ is the gravitational mass of a central gravitating body and $q$ represents the electric charge,

$$\alpha = \text{Re}(\xi), \qquad \lambda = (\lambda_1, \lambda_2, \lambda_3) \tag{3.15}$$

$$\lambda = -\frac{2\,\text{Re}(\xi^2\nabla\xi^*) + i\nabla\xi \times \nabla\xi^*}{(\xi\xi^*)^2 + \nabla\xi\cdot\nabla\xi^*} \tag{3.16}$$

$$\xi = \frac{1}{(r^2 - \gamma + 2i\sqrt{\gamma}\,x^3)^{1/2}} \tag{3.17}$$

and

$$r = [(x^1)^2 + (x^2)^2 + (x^3)^2]^{1/2} \tag{3.18}$$

$\gamma$ is a real parameter related to the angular momentum per unit mass $h$ by the relation $h^2 = \gamma - g^2 + l^2$, where $g$ and $l$ can be interpreted as the magnetic charge (monopole) and the magnetic mass (NUT) parameters, respectively.

The metric in which we are interested is generated in the same way as the Kerr–Newman metric in NGR is generated. The only difference is that the angular momentum per unit mass parameter $h$ in the Kerr–Newman metric will be replaced according to the relation $h^2 = \gamma - g^2 - l^2$. The justification behind using the relation $\gamma - g^2 + l^2$ in the place of $h^2$ is that the MR solution in GR can be generated from the Kerr–Newman solution in GR with the replacement of $h^2$ by $\gamma - g^2 + l^2$ and $\Sigma = \rho^2 + h^2\cos^2\theta$ in the Kerr–Newman metric by

$$\Sigma = \gamma^2 + (1 - \sqrt{\gamma}\cos\theta)^2$$

With the coordinate transformation from $(x^\mu)$ to $(t, \rho, \theta, \phi)$ defined by

$$x^0 = t + \frac{a}{2}\ln|\Delta| + \frac{a^2}{2}\left(1 - \frac{e^2}{a^2}\right)B$$

$$x^1 = (\rho\cos\Phi + \sqrt{\gamma}\sin\Phi)\sin\theta$$

$$x^2 = (\rho\sin\Phi - \sqrt{\gamma}\cos\Phi)\sin\theta$$

$$x^3 = \rho\cos\theta$$

$$\phi = \Phi + \sqrt{\gamma}B \tag{3.19}$$

where

$$\rho = \left\{\frac{r^2 - \gamma}{2} + \left[\left(\frac{r^2 - \gamma}{2}\right)^2 + (\sqrt{\gamma}x^3)^2\right]^{1/2}\right\}^{1/2} \tag{3.20}$$

$$\Delta = \rho^2 + \gamma - a\rho + e^3 \tag{3.21}$$

$$B = \int^{\rho} \frac{d\rho}{\Delta} \tag{3.22}$$

the metric and the electromagnetic potential can be written as

$$\begin{aligned} ds^2 = & \frac{\gamma \sin^2\theta - \Delta}{\Sigma} dt^2 \\ & + \frac{2\sqrt{\gamma}(\rho^2 + \gamma - \Delta)\sin^2\theta}{\Sigma} dt\, d\phi \\ & + \frac{(\rho^2 + \gamma)^2 - \Delta\gamma \sin^2\theta}{\Sigma} \sin^2\theta\, d\phi^2 \\ & + \frac{\Sigma}{\Delta} d\rho^2 + \Sigma\, d\theta^2 \end{aligned} \tag{3.23}$$

and

$$\begin{aligned} A &= A_t\, dt + A_\rho\, d\rho + A_\theta\, d\theta + A_\phi\, d\phi \\ &= -\frac{q\rho}{4\pi\Sigma}(dt + \sqrt{\gamma}\sin^2\theta\, d\phi) \end{aligned} \tag{3.24}$$

where

$$\Sigma = \rho^2 + (1 - \sqrt{\gamma}\cos\theta)^2 \tag{3.25}$$

To obtain expression (3.24) from $A_\mu$ of equation (3.12) we used a $U(1)$ gauge transformation. The parallel vector fields $b^k_\mu$ are expressed as

$$\begin{aligned} b^0_t &= 1 - \frac{a\rho - e^2}{2\Sigma} \\ b^0_\rho &= \frac{a\rho - e^2}{2\Delta} \\ b^0_\theta &= 0 \\ b^0_\phi &= \frac{\sqrt{\gamma}(a\rho - e^2)\sin^2\theta}{2\Sigma} \\ b^1_t &= \frac{a\rho - e^2}{2\Sigma}\sin\theta\cos\Phi \\ b^1_\rho &= \frac{\sin\theta}{\Delta}\left(\rho X - \frac{a\rho - e^2}{2}\cos\Phi\right) \end{aligned}$$

$$
\begin{aligned}
b^1_\theta &= X\cos\theta\\
b^1_\phi &= -Y\sin\theta + \frac{\sqrt{\gamma}(a\rho - e^2)}{2\Sigma}\sin^3\theta\cos\Phi\\
b^2_t &= \frac{a\rho - e^2}{2\Sigma}\sin\theta\sin\Phi\\
b^2_\rho &= \frac{\sin\theta}{\Delta}\left(\rho Y - \frac{a\rho - e^2}{2}\sin\Phi\right)\\
b^2_\theta &= Y\cos\theta\\
b^2_\phi &= X\sin\theta + \frac{\sqrt{\gamma}(a\rho - e^2)}{2\Sigma}\sin^3\theta\sin\Phi\\
b^3_t &= \frac{a\rho - e^2}{2\Sigma}\cos\theta\\
b^3_\rho &= \left(1 + \frac{a\rho - e^2}{2\Delta}\right)\cos\theta\\
b^3_\theta &= -\rho\sin\theta\\
b^3_\phi &= \frac{\sqrt{\gamma}(a\rho - e^2)}{2\Sigma}\sin^2\theta\cos\theta
\end{aligned}
\tag{3.26}
$$

where $X$ and $Y$ are given by

$$
\begin{aligned}
X &= \rho\cos\Phi + \sqrt{\gamma}\sin\Phi\\
Y &= \rho\sin\Phi - \sqrt{\gamma}\cos\Phi
\end{aligned}
\tag{3.27}
$$

Thus a solution of NGR is obtained. In GR this solution corresponds to

$$
(e^k_\mu) =
\begin{bmatrix}
\left(1 - \frac{a\rho - e^2}{\Sigma}\right)^{1/2} & 0 & 0 & -\frac{\sqrt{\gamma}(a\rho - e^2)\sin^2\theta}{\Sigma - a\rho + e^2}\left(1 - \frac{a\rho - e^2}{\Sigma}\right)^{1/2}\\
0 & \left(\frac{\Sigma}{\Delta}\right)^{1/2} & 0 & 0\\
0 & 0 & \sqrt{\Sigma} & 0\\
0 & 0 & 0 & \left(\frac{\Delta\Sigma}{\Sigma - a\rho + e^2}\right)^{1/2}\sin\theta
\end{bmatrix}
\tag{3.28}
$$

The metric given in (3.28) is related to $b^k_\mu$ of equation (3.27) through a local Lorentz transformation. Hence we have obtained a solution of NGR from a solution of GR by choosing a local Lorentz transformation such that the axial vector part of the torsion tensor made of the parallel vector field vanishes.

The solution given by (3.28) corresponds to:

(i) Kerr–Newman–Kasuya spacetime when $l = 0$.
(ii) Kerr–Newman spacetime for $g = l = 0$.
(iii) Kerr spacetime in the case of $e = g = l = 0$.
(iv) NUT spacetime when $\gamma = e = g = 0$.

## 4. REMARKS

From this work it appears that any solution of GR having a Killing horizon can be transformed into a solution of NGR.

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